Peper No. XI
classical mechanics
classical electrodynamics

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Unit I :- Lagrangian Formulation.

Constraints :-

1) Holonomic constraints :-

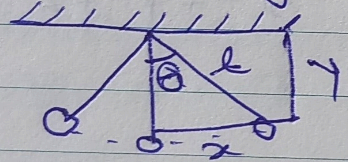
The Constraints which follow the eq^n

$$f(\vec{r_1}, \vec{r_2}, \vec{r_3} \cdots \vec{r_n}, t) = 0$$

$r_1, r_2, r_3 \cdots$ are Co-ordinates
t is function.

are called as Holonomic Constraints
e.g. simple pendulum, motion of rigid body

e.g.



$$x = l\sin\theta$$
$$y = l\cos\theta$$

$$x^2+y^2 = l^2(\sin^2\theta + \cos^2\theta)$$

$$x^2+y^2 = l^2$$

this term we can Write in terms of equality
they are independent of velocity

2) Non - Holonomic constraints :-

The constraints which do not follow the eqn.

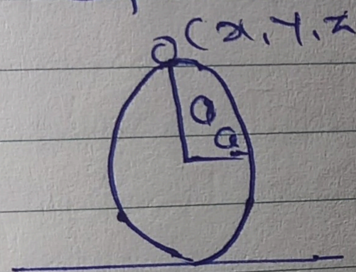
$$f(\vec{r_1}, \vec{r_2}, \vec{r_3} \cdots \vec{r_n}, t) = 0$$

or they have inequality as.

$$f(\vec{r_1}, \vec{r_2}, \vec{r_3} \cdots \vec{r_n}, t) \neq 0$$

Those constraints are called non. Holonomic constraints.

eg. motion of partical from the top of sphere.



$$x^2+y^2+z^2 \geq a^2$$

The non - Holonomic constraints can be written as inequality.

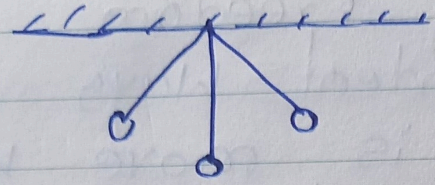
these constraints are dependent on velocity of partical

Holonomic :- types

i) scleronomous constraints :- No depend on time.

The constraints which do not contain time explicitly is called scleronomous

for. eg. motion of simple pendulum



(x, y)

$l^2 = x^2 + y^2$

eqn of motion do not contain 't' as explicit variable

ii) Rheonomous constraints :-

Those constraints which contains time t as a explicit, variable are called as Rheonomous

e.g 1) motion of partical along the plane wave

2) Motion of pendulum whose length changes with time

e.g. $l = l(t)$

3) Radius of sphere changing with time. $a = a(t)$

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Degrees of freedom

The no. of independent ways in which mechanical system is move without violating any constraints which may be imposed on the system is called No. of degrees of freedom.

$$D.O.F = 3N - K$$

$K \rightarrow$ No. of constraints of motion

In case of Holonomic constraints the difficulty that the co-ordinates $\vec{r}$ are no longer independent. Hence the eq^n of motion are not all independent and is solved by generalise co-ordinates a system of N particals free from constraints has 3N independent co-ordinates or degrees of freedom. If there exists Holonomic constraints expreesed in K eq^n of the form

$$f(\vec{r}_1, \vec{r}_2 \ldots \vec{r}_n, t) = 0$$

Then we use these eq^n to eliminate K out of 3N co-ordinates & we left 3N-K degrees of freedom. Independent variables $q_1, q_2, q_3 \ldots q_{3N-K}$ are introduced to eliminate dependent co-ordinates $r_1, r_2, r_3 \ldots \vec{r}_n$. The old co-ordinates can be expressed in terms of new co-ordinates by the eq^n of the form

$$\vec{r}_1 = \vec{r}_1 (q_1, q_2 \cdots q_{3N-k}, t)$$
$$\vec{r}_2 = \vec{r}_2 (q_1 . q_2 \cdots q_{3N-k}, t)$$
$$\vdots$$
$$\vec{r}_n = \vec{r}_n (q_1 , q_2 \cdots q_{3N-k} , t)$$

this transformation eq^N from the set of $\vec{r}_i$ variables to the q_i set the inverse transformation eq^N given by

$$q_1 = q_1 (\vec{r}_1 , \vec{r}_2 \cdots \vec{r}_N , t)$$
$$\vdots$$
$$q_{3N-k} = q_{3N-k} (\vec{r}_1 , \vec{r}_2 \cdots \vec{r}_N , t)$$

The time derivative of generlise co-ordinates $\left| \dot{q}_j = \frac{\partial q_j}{\partial t} \right|$ are called generalised velocity

generalised co-ordinates q_j may not necessary Have the dimention of length. ~~for~~ ~~to~~

for to choose suitable set of generalise co-ordinates following three principles.

1) These values should determined ~~by~~ the configuration of the system.

2) They may be varied arbitarily & independently of each other without violating the constraints on them

3) There is no uniquness in the ~~co~~ choise of generalised co-ordinates the choice should fall on the set of

co-ordinates that is give a resionable mathematical simplification of the problem.

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Principle of virtual work

Let an infinitesimal displacement δr_i of the system & let f_i will be total forces acting on the system.

$$\text{Workdone} = \sum_i f_i \delta r_i$$

If the system is in equilibrium

$$\sum_i f_i \delta r_i = 0 \quad — (1)$$

The forces vanishes when system undergo equilibrium.

The forces f_i can be written as

$$f_i = f_i^{(a)} + f_i$$

force of constraint

Then eq^n (1) becomes

$$\sum_i f_i \delta r_i = 0$$

$$\sum_i (f_i^{(a)} + f_i) \delta r_i = 0$$

$$\sum_i f_i^{(a)} \delta r_i + \sum_i f_i \delta r_i = 0$$

We assume that forces of constraint are zero.

or we restrict the system to forces of constraint are zero

$$\sum_i f_i \, \delta r_i = 0$$

$$\sum_i f_i^{(a)} \, \delta r_i = 0$$

This eqⁿ is known as principle of virtual work.

It states that the virtual work done by the applied forces acting on a system in equilibrium is zero, provided that no frictional forces are present.

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D-Alembert's principle :-

We have the eqⁿ in terms of force

$$\vec{f_i} = \dot{\vec{p_i}}$$

$$f_i - \dot{\vec{p_i}} = 0 \qquad — \textcircled{1}$$

According to principle of virtual work

$$\sum_i \left(f_i^{(a)} - \dot{\vec{p_i}}\right) \delta r_i = 0$$

If system is restricted to constraints

$$\sum \left(f_i^{(a)} - \dot{\vec{p_i}}\right) \delta r_i + \sum_i f_i \, \delta r_i = 0$$

This eq^N is known as D-Alembert's principle.
Where f_i are forces of constraints
We have co-ordinate transformations

$$\vec{r_i} = \vec{r_i}(q_1, q_2, \ldots q_3, t)$$

virtual displacement δr_i is given by

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j} \delta q_j \quad - (2)$$

from eq^N (1)

$$\sum_i (f_i - \dot{\vec{p_i}}) \delta r_i = 0$$

$$\therefore \sum_i f_i \delta r_i = \sum_i \dot{\vec{p_i}} \delta r_i \quad - (3)$$

Now $L.H.S = \sum_i f_i \delta r_i$

from eq^N (2)

$$\sum_i f_i \delta r_i = \sum_j \left(\sum_i f_i \frac{\partial r_i}{\partial q_j} \right) \delta q_j$$

$$L.H.S = \sum_j Q_j \delta q_j \quad - (4)$$

where $Q_j = \sum_i f_i \frac{\partial r_i}{\partial q_j}$ are called as Generalised force

Now. $\vec{p_i} = \frac{d}{dt}(m_i \dot{\vec{r_i}})$

$$= m_i \ddot{\vec{r_i}}$$

- J. R. Lowell

$$R.H.S = \sum_i \vec{p_i}\, \delta r_i$$

$$= \sum_i m_i \vec{\ddot{r}_i}\, \delta r_i$$

$$= \sum_j \sum_i m_i \left(\vec{\ddot{r}_i} \frac{\partial r_i}{\partial q_j} \right) \delta q_j \quad \text{— (5)}$$

$$\frac{d}{dt}\left(\vec{\dot{r}} \frac{\partial r_i}{\partial q_j} \right) = \vec{\ddot{r}_i} \frac{\partial r_i}{\partial q_j} + \vec{\dot{r}} \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right)$$

$$\vec{\ddot{r}} \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\vec{r}} \cdot \frac{\partial r_i}{\partial q_j} \right) - \vec{\dot{r}} \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right)$$

$$= \frac{d}{dt}\left(\vec{\dot{r}} \frac{\partial r_i}{\partial q_j} \right) - \vec{\dot{r}} \frac{\partial \vec{\dot{r}}}{\partial q_j} \Rightarrow$$

Interchanging der q_j & t

Taking derivative w.r.t. time b.s.

$$= \frac{d}{dt}\left(\vec{\dot{r}} \cdot \frac{\partial \vec{\dot{r}_i}}{\partial q_j} \right) - \vec{\dot{r}} \frac{\partial \vec{\dot{r}_i}}{\partial q_j}$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial q_j}\left(\frac{1}{2} \vec{\dot{r}_i}^{\,2} \right) \right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2} \vec{\dot{r}_i}^{\,2} \right) \quad \text{— (6)}$$

Put eq^N (6) in eq^N (5)

$$R.H.S = \sum_j \sum_i m_i \left[\frac{d}{dt}\left(\frac{\partial}{\partial q_j}\left(\frac{1}{2} \vec{\dot{r}_i}^{\,2} \right) \right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2} \vec{\dot{r}_i}^{\,2} \right) \right] \partial q_j$$

$$= \sum_j \left(\frac{d}{dt}\left(\frac{\partial}{\partial q_j} \sum_i \frac{1}{2} m_i \vec{\dot{r}_i}^{\,2} \right) - \frac{\partial}{\partial q_j} \sum_i \frac{1}{2} m_i \vec{\dot{r}_i}^{\,2} \right) \delta q_j$$

but we know. $T = \frac{1}{2} m v^2 = \frac{1}{2} m_i \dot{\vec{r}_i}^{\,2}$

$$= \sum_j \left(\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_j} (T) \right] - \frac{\partial}{\partial q_j} (T) \right] \delta q_j$$

$$= \sum_j \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j \quad - (7)$$

$$\therefore \sum_j \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j = \sum_j Q_j \delta q_j$$

$$\sum_j \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0 \quad - (8)$$

This eq^N is known as Euler-lagranges eq^N

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* Lagrange's eq^N from D-Alembert's principle :-

We have D-Alembert's eq^N in terms of Generalised co-ordinates

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = Q_j \delta q_j \quad - (1)$$

but we have.

$$Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j}$$

but for conservative system.

$$\vec{\nabla} \times \vec{F} = 0$$

$$\vec{F} = -\vec{\nabla} \cdot V$$

where V is the potential

then. $Q_j = \sum_i -\nabla V \frac{\partial r_i}{\partial q_j}$

but $V = V(\vec{r_1}, \vec{r_2} \cdots \vec{r_n} \cdot t)$

$$\frac{\partial V}{\partial q_j} = \sum_i \vec{\nabla} V \frac{\partial r_i}{\partial q_j} \quad -(2)$$

put eq^n (2) in eq^n (1)

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j = Q_j \delta q_j$$

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = -\frac{\partial V}{\partial q_j} \delta q_j \quad -(3)$$

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} \right] \delta q_j = 0$$

Now q_j are the set of independent co-ordinates which are independent of q_K. so, $\delta r_i \neq 0$

then, $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) = \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

We have $V = V(q_j)$

$$\frac{dV}{\partial \dot{q}_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0 \quad - (5)$$

We define $T - V = L$
where $L \rightarrow$ is Lagrangian
put in eq^n (4)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

This eq^n is known as lagrange's eq^n of motion.

In Newtonian We were deal with forces Where in lagrangian mechanics We deal with Energies.

In newtonian mechanics we study acceleration. momentum. force. where in lagrangian we deal with K.E & P.E only

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Application of Lagranges eq^N

for ~~cartesion~~
Lagranges eq^N of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

D-Alemberts principle of Generalised co-ord
~~for cartesion co-ordinate~~

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

for Cartesion co-ordinate system

$$x \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$y \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$z \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

for sperical polar co-ordinate system

$$r \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

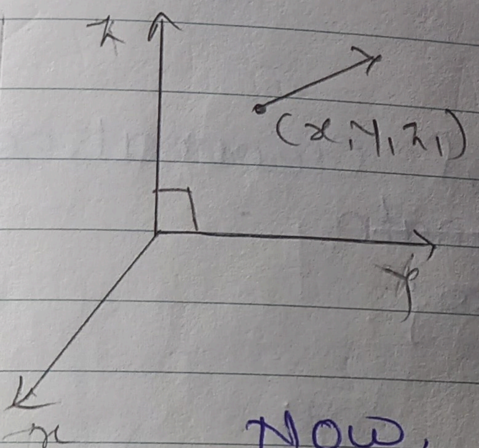
$$\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

8 $\phi = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$

Appl^n

9 i) Motion of partical in space using cartesion Co-ordinates:-

10



11 Consider a partical moving in plane The Co-ordinate associated

12 With partical are x, y, z resp. along x, y & z-axis

1 Now, $\dot{x}, \dot{y}$ & $\dot{z}$ are their velocities along resp. axis.

2 If m is the mass of partical then.

3 Kinetic energy $T = \frac{1}{2} mv^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

4 But consider f_x, f_y, f_z are forces acting on the partical in x, y, & z direction. resp.

5 Now, from D-Alembert's principle of Generalised forces.

6 $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{--- (1)}$$

7 $$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Now $\frac{\partial T}{\partial \dot{x}} = \frac{1}{2} m (2\dot{x}) = m\dot{x}$

& $\frac{\partial T}{\partial x} = 0$ putting this values in eqn ①

We get

$$\frac{d}{dt}(m\dot{x}) - 0 = f_x$$

$$m\ddot{x} = f_x$$

$$f_x = m\ddot{x}$$

for y co-ordinate

$$\frac{\partial T}{\partial \dot{y}} = m\dot{y} \quad \& \quad \frac{\partial T}{\partial y} = 0$$

$$\frac{d}{dt}(m\dot{y}) - 0 = f_y$$

$$m\ddot{y} = f_y$$

$$f_y = m\ddot{y}$$

for z co-ordinate

$$\frac{\partial T}{\partial \dot{z}} = m\dot{z} \quad \& \quad \frac{\partial T}{\partial z} = 0$$

$$\frac{d}{dt}(m\dot{z}) - 0 = f_z$$

$$m\ddot{z} = f_z$$

$$f_z = m\ddot{z}$$

$\therefore$ $f_x = m\ddot{x}$ $\quad f_y = m\ddot{y}$ $\quad f_z = m\ddot{z}$

There are forces in x, y, z direction resp.

Acclⁿ of the partical is given by $\frac{F}{m}$ along x direction it will be $F_x/m = \ddot{x}$ Along y direction
$A_y = F_y/m = \ddot{y}$ & Along z direction
Accelⁿ $A_z = F_z/m = \ddot{z}$.

Application2 :- Atwood machine :-
Newtonian :- forces acting along the masses are $m_1 g$ & $m_2 g$ resp. ~~then~~.

$$m_2 g - T = m_2 a$$
$$T - m_1 g = m_1 a$$
$$m_2 g - m_1 g = (m_1 + m_2) a$$

$$(m_1 - m_2) g = (m_1 + m_2) a$$

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g.$$

21/1/25 consider massless, frictionless pully & two masses m_1 & m_2 are attached to it. The length of string is fixed i.e l This is holonomic stéronomous constraint. Then K.E. of the system is given by

for $m_1 = \frac{1}{2} m_1 \dot{x}^2$
for $m_2 = \frac{1}{2} m_2 (\dot{l} - \dot{x})^2 = \frac{1}{2} m_2 \dot{x}^2$

Total K.E. $= \frac{1}{2} (m_1 + m_2) \dot{x}^2$ — (1)

Then P.E V for $m_1 = -m_1 g x$

for mass $m_2 = -m_2 g(l-x)$

8 Total P.E $= -m_1 g x - m_2 g(l-x)$ — (2)

9 Then Lagrangiean

$$L = T - V$$

10 $$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + [m_1 g x + m_2 g(l-x)]$$

11 Now we have

Lagrange's eq^n of motion as

12 $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{— (3)}$$

1 Now $$\frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} \quad \text{— (4)}$$

2 $$\frac{\partial L}{\partial x} = m_1 g - m_2 g$$

3 $$\frac{\partial L}{\partial x} = (m_1-m_2)g \quad \text{— (5)}$$

4 $$\frac{d}{dt}[(m_1+m_2)\dot{x}] - (m_1-m_2)g = 0$$

5 $\ddot{x} = (m_1 -$

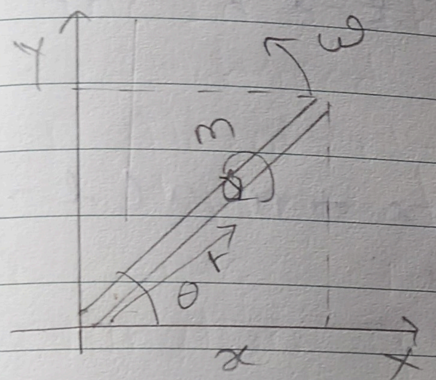
$$(m_1+m_2)\ddot{x} = (m_1-m_2)g$$

6

7 $$\ddot{x} = \frac{(m_1+m_2)}{(m_1+m_2)} g$$

This is acel^n of the masses of the direction of Gravitational fields.

~~3)~~ Beads moving on uniformly rotating wire under force free condn :-



Since the wire is rotating under force free Condn.
The potential energy of bead $V = 0$
Then from fig.

fig. beads moving along uniformly rotating Wire.

$x = r\cos\theta = r\cos\omega t$ —(1)
$y = r\sin\theta = r\sin\omega t$ —(2)

We know

$$K.E = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

Now,

$$\dot{x} = \frac{dx}{dt} = \frac{dr}{dt}\cos\omega t - r\omega\sin\omega t \quad \text{—(3)}$$

$$\dot{y} = \frac{dy}{dt} = \frac{dr}{dt}\sin\omega t + r\omega\cos\omega t \quad \text{—(4)}$$

$$\dot{x}^2 = \left(\frac{dr}{dt}\right)^2 \cos^2\omega t + r^2\omega^2\sin^2\omega t - 2r\omega\left(\frac{dr}{dt}\right)\sin\omega t \cdot \cos\omega t$$

$$\dot{y}^2 = \left(\frac{dr}{dt}\right)^2 \cdot \sin^2\omega t + r^2\omega^2\cos^2\omega t \; \cancel{+2r\omega} + 2r\omega\left(\frac{dr}{dt}\right)\sin\omega t\cos\omega t$$

$$\dot{x}^2 + \dot{y}^2 = \left(\frac{dr}{dt}\right)^2 (\cos^2\omega t + \sin^2\omega t) + r^2\omega^2(\cos^2\omega t + \sin^2\omega t)$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\omega^2$$

$$= \dot{r}^2 + r^2\omega^2 \quad - (3)$$

Then $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$$= \frac{1}{2} m (\dot{r}^2 + r^2\omega^2)$$

$$L = T - V$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\omega^2) - 0$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\omega^2)$$

We have lagrangian eq^n of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = m\omega^2 r$$

$$\frac{d}{dt}(m\dot{r}) - mr\omega^2 = 0$$

$$m\ddot{r} - mr\omega^2 = 0$$

$$\ddot{r} - r\omega^2 = 0$$

$$\ddot{r} = r\omega^2$$

This is centripetal accelerating on the bead due to circular motion.